POSITIVE DEHN TWIST EXPRESSIONS FOR SOME NEW INVOLUTIONS IN MAPPING CLASS GROUP

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ABSTRACT. The well-known fact that any genus g symplectic Lefschetz fibration $X^4 \rightarrow S^2$ is given by a word that is equal to the identity element in the mapping class group and each of whose elements is given by a positive Dehn twist, provides an intimate relationship between words in the mapping class group and 4-manifolds that are realized as symplectic Lefschetz fibrations. In this article we provide new words in the mapping class group, hence new symplectic Lefschetz fibrations. We also compute the signatures of those symplectic Lefschetz fibrations.

INTRODUCTION

The last decade has experienced a resurgence of interest in the mapping class groups of 2-dimensional closed oriented surfaces. This is primarily due to Donaldson's theorem [3] that any closed oriented symplectic 4−manifold has the structure of a Lefschetz pencil and Gompf's theorem [5] that any Lefschetz pencil supports a symplectic structure. These theorems, together with the fact that any genus g symplectic Lefschetz fibration is given by a word that is equal to the identity element in the mapping class group and each of whose elements is given by a positive Dehn twist, provides an intimate relationship between words in the mapping class group and 4−dimensional symplectic topology. Therefore the elements of finite order in the mapping class group are of special importance. There are very few examples of elements of finite order for which explicit positive Dehn twist products are known and a great deal is known about the structure of the 4−manifolds that they describe. In this article we give an algorithm for positive Dehn twist products for a set of new involutions in the mapping class group that are non-hyperelliptic. These involutions are obtained by combining the positive Dehn twist expressions for two well-known involutions of the mapping class group.

An application of the words in the mapping class group that we produced is to determine the homeomorphism types of the 4−manifolds that they describe. This calculation involves computing two invariants of those manifolds. The first one, which is very easy to compute, is the Euler characteristic, and the second one is the signature. Using the algorithm described in [17] we wrote a Matlab program that computes the signatures of some of these manifolds. The computations that we have done using this program point to a closed formula for the signature, which is mentioned in the last part of this article.

This article will be followed by two other articles, the first one of which will contain the same computations for the multiple case. Namely, the explicit positive

1991 *Mathematics Subject Classification.* Primary 57M07; Secondary 57R17, 20F38.
Key words and phrases. low dimensional topology, symplectic topology, mapping class group, Lefschetz fibration .

Dehn twist products for the involutions obtained by combining several involutions of the type described in this article and the signature computations for the 4-manifolds they describe as symplectic Lefschetz fibrations will be presented. The third article will contain the same work for some finite order elements in the mapping class group that are rotations through $2\pi/p, p-$odd.

1. Mapping Class Groups

1.1. **Definition of M_g.** Let Σ_g be a 2-dimensional, closed, compact, oriented surface of genus $g > 0$. From now on, g will always be positive unless otherwise stated. Let $Diff^+(\Sigma_g)$ be the group of all orientation-preserving diffeomorphisms $\Sigma_g \to \Sigma_g$, and $Diff_0^+(\Sigma_g)$ be the subgroup of $Diff^+(\Sigma_g)$ consisting of all orientation-preserving diffeomorphisms $\Sigma_g \to \Sigma_g$ that are isotopic to the identity.

The mapping class group M_g of Σ_g is defined to be the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g, i.e.,

$$M_g = Diff^+(\Sigma_g)/Diff_0^+(\Sigma_g).$$

1.2. **Generators.**

Definition 1.2.1. Let α be a simple closed curve on Σ_g. Cut the surface Σ_g open along α and glue the ends back after rotating one of the ends $360°$ to the right, which makes sense if Σ_g is oriented. This operation defines a diffeomorphism $\Sigma_g \to \Sigma_g$ supported in a small neighborhood of α and *the positive Dehn twist about* α is defined to be the isotopy class of this diffeomorphism. It is denoted by t_α or $D(\alpha)$.

Fact 1. t_α is independent of how α is oriented.

A simple closed curve α on Σ_g is called *nonseparating* if $\Sigma_g \backslash \alpha$ has one connected component and it is called *separating* otherwise.

Fact 2. M_g is generated by Dehn twists about nonseparating simple closed curves.

This fact was first proven by Dehn in 1938 [2]. His set of generators consisted of $2g(g-1)$ generators. Much later (1964) Lickorish independently proved that $3g-1$ Dehn twists are enough to generate M_g [10]. Humphries (1979) proved that $2n+1$ Dehn twists are enough and this is the minimal number of twists to generate M_g [6]. If we do not require the generators to be Dehn twists then the mapping class groups can be generated by 2 elements [19].

This is clearly the lowest number of generators as we know that M_g is not an abelian group.

1.3. **Presentation.** First, we need a few technical lemmas that will be useful in proving some of the defining relations in the presentation for M_g.

Lemma 1.3.1. *Let α and β be two non-separating simple closed curves on Σ_g. Then there is a diffeomorphism $f : \Sigma_g \to \Sigma_g$ such that $f(\alpha) = \beta$.*

For a proof see [7].

Let $\mathcal{S}(\Sigma_g)$ be the set of all isotopy classes of simple closed curves in Σ_g. For $[\alpha]$, $[\beta] \in \mathcal{S}(\Sigma_g)$ define $I([\alpha],[\beta]) = \min\{|a \cap b| \; |a \in [\alpha], b \in [\beta]\}$.

Definition 1.3.2. If two simple closed curves α and β intersect each other transversely at one point, denote it by $\alpha\perp\ \beta$, then we define the product $\alpha\beta$ as the simple closed curve obtained from $\alpha\cup\beta$ by resolving the intersection point $\alpha\cap\beta$ according to Figure 1.

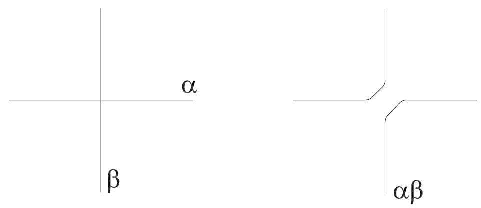


FIGURE 1. Resolving an Intersection

If $|\alpha\cap\beta| = I\left([\alpha],[\beta]\right) = k$ then $\alpha\beta$ is defined as the simple closed curve obtained from $\alpha\cup\beta$ by resolving each of the k intersection points according to Figure 1.
Notation: $\alpha(\alpha\beta) = \alpha^2\beta$, $\alpha(\alpha(\alpha\beta)) = \alpha^3(\beta), \ldots, \alpha(\alpha\cdots(\alpha\beta)) = \alpha^k\beta$.

Remark 1. (*a*) If $I\left([\alpha],[\beta]\right) = 1$, then $t_\alpha\left(\beta\right) = \alpha\beta$ and $t_\beta\left(\alpha\right) = \beta\alpha$.
(*b*) If $I\left([\alpha],[\beta]\right) = 0$, then $t_\alpha\left(\beta\right) = \beta$ and $t_\beta\left(\alpha\right) = \alpha$.

Lemma 1.3.3. (*a*) *If α and β are two simple closed curves, then $t_\alpha\left(\beta\right) = \alpha^k\beta$, where $I\left([\alpha],[\beta]\right) = k$, ([12]).*
(*b*) $I\left([\alpha],[\beta]\right) = I\left([\alpha\beta],[\beta]\right) = I\left([\alpha\beta],[\alpha]\right)$.

Lemma 1.3.4. *If $I\left([\alpha],[\beta]\right) = 1$, then $\alpha\left(\beta\alpha\right) = \beta$ and $\beta\left(\alpha\beta\right) = \alpha$, namely, $t_\alpha t_\beta\left(\alpha\right) = \beta$ and $t_\beta t_\alpha\left(\beta\right) = \alpha$.*

Proof: The following figure is a sketch of the proof.

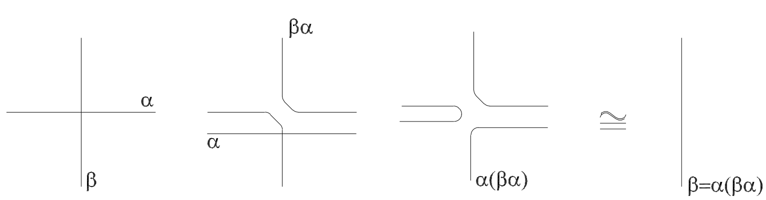


FIGURE 2. $t_\alpha t_\beta\left(\alpha\right) = \beta$ and $t_\beta t_\alpha\left(\beta\right) = \alpha$, for $I\left([\alpha],[\beta]\right) = 1$

Lemma 1.3.5. *For any simple closed curve α in Σ_g and any diffeomorphism f : $\Sigma_g \to \Sigma_g$, we have $t_{f(\alpha)} = f t_\alpha f^{-1}$.*

For a proof see [7].

Corollary 1.3.6. *The Dehn twists about any two nonseparating simple closed curves α and β are conjugate, i.e., $t_\beta = f t_\alpha f^{-1}$, for some $f : \Sigma_g \to \Sigma_g$.*

This easily follows from the previous lemma and Lemma 1.3.1.

Remark 2. (*a*) In particular if $f = t_\beta$ in Lemma 1.3.5, then $t_{t_\beta(\alpha)} = t_\beta t_\alpha t_\beta^{-1}$.
(*b*) Moreover, if $I([\alpha], [\beta]) = 1$, then $t_{\beta\alpha} = t_\beta t_\alpha t_\beta^{-1}$, because $t_\beta(\alpha) = \beta\alpha$ by Remark 1 (a) .

Commutativity and Braid Relations

Lemma 1.3.7. (*a*) *If $I([\alpha], [\beta]) = 0$, then $t_\alpha t_\beta = t_\beta t_\alpha$.*
(*b*) *If $I([\alpha], [\beta]) = 1$, then $t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta$.*

Proof:

(*a*) We have $t_\beta(\alpha) = \alpha$ by Remark 1 (b) . Therefore, $t_{t_\beta(\alpha)} = t_\beta t_\alpha t_\beta^{-1}$ in Remark 2 (a) becomes $t_\alpha = t_\beta t_\alpha t_\beta^{-1}$.

(*b*) Since $I([\alpha], [\beta]) = 1$ we have $I([\alpha], [\beta\alpha]) = 1$ by Lemma 1.3.3 (b). Therefore, we also have $t_\alpha(\beta\alpha) = \alpha(\beta\alpha)$ using Remark 1 (a), which is equal to β, using Lemma 1.3.4. Now, the twist along β can be expressed as

$$t_\beta = t_{t_\alpha(\beta\alpha)} = t_\alpha t_{\beta\alpha} t_\alpha^{-1},$$

thanks to Lemma 1.3.5. Finally substituting $t_\beta t_\alpha t_\beta^{-1}$ for $t_{\beta\alpha}$ in the last equation, using Remark 2 (b), we get

$$t_\beta = t_\alpha t_\beta t_\alpha t_\beta^{-1} t_\alpha^{-1},$$

which finishes the proof.

The relation $t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta$, for $[\alpha] \neq [\beta]$, is called the *braid relation* and we say t_α and t_β are *braided* if they satisfy the braid relation.

The following formula plays the key role in proving the converse of Lemma 1.3.7, ([4]).

Formula 1. For two simple closed curves α and β we have

$$I([t_\alpha^n(\beta)], [\beta]) = |n| \, I([\alpha], [\beta])^2 .$$

In addition to Formula 1, the following two simple facts will also be used in proving the converse of Lemma 1.3.7.

Fact 3. For two simple closed curves α and β, if $[\alpha] \neq [\beta]$, then $t_\alpha \neq t_\beta$, ([13]).

Fact 4. For two simple closed curves α and β if $I([\alpha], [\beta]) \neq 0$, then $[t_\alpha(\beta)] \neq [\beta]$.

Proof: Formula 1 for $n = 1$ gives $I([t_\alpha(\beta)], [\beta]) = I([\alpha], [\beta])^2 \neq 0$. If $[t_\alpha(\beta)] = [\beta]$ then $I([t_\alpha(\beta)], [\beta]) = I([\beta], [\beta])$. Since $I([\beta], [\beta]) = 0$ we can't have $[t_\alpha(\beta)] = [\beta]$.

Lemma 1.3.8. (*a*) *For two simple closed curves α and β if $t_\alpha t_\beta = t_\beta t_\alpha$, then $I([\alpha], [\beta]) = 0$.*
(*b*) *For two simple closed curves α and β, with $[\alpha] \neq [\beta]$, if $t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta$ then $I([\alpha], [\beta]) = 1$.*

Proof:

(a) From $t_\alpha t_\beta = t_\beta t_\alpha$ we get $t_\alpha t_\beta t_\alpha^{-1} = t_\beta$. Using Lemma 1.3.5 we conclude that $t_{t_\alpha(\beta)} = t_\beta$. So, $[t_\alpha(\beta)] = [\beta]$ by Fact 3 and therefore $I([\alpha],[\beta]) = 0$ using Fact 4.

(b) From $t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta$ we get $t_\alpha t_\beta t_\alpha (t_\alpha t_\beta)^{-1} = t_\beta$. Using Lemma 1.3.5 we get $t_{t_\alpha t_\beta(\alpha)} = t_\beta$, and therefore $[t_\alpha t_\beta(\alpha)] = [\beta]$ by Fact 3. Applying Formula 1 with $n = 1$ we get

$$I([\alpha],[\beta])^2 = I([t_\beta(\alpha)],[\alpha]) = I([t_\alpha t_\beta(\alpha)],[\alpha]) = I([\beta],[\alpha]) = I([\alpha],[\beta]).$$

Therefore $I([\alpha],[\beta])$ is 0 or 1. If it is 0, then t_α and t_β commute by Lemma 1.3.7 and $t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta$ becomes $t_\beta t_\alpha^2 = t_\beta^2 t_\alpha$, and therefore $t_\alpha = t_\beta$, i.e., $[\alpha] = [\beta]$, which is a contradiction.

For more results on the previous two Lemmas see [15].

Lantern and Chain Relations

The following two relations will be used for the presentation of M_g, along with the commutativity and braid relations.

Let $\Sigma_{0,4}$ be a sphere with four holes. If c_1, c_2, c_3, and c_4 are the boundary curves of $\Sigma_{0,4}$ and the simple closed curves α and β are as shown in Figure 3 , then we have

$$t_\alpha t_\beta t_{\alpha\beta} = t_{c_1} t_{c_2} t_{c_3} t_{c_4},$$

where t_{c_i}, $1 \leq i \leq 4$, denote the Dehn twists about c_i.

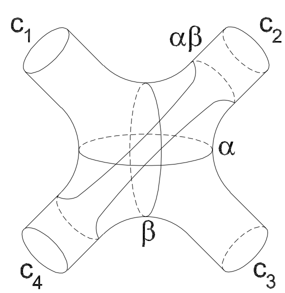


Figure 3. The Lantern Relation

This relation was known to Dehn and later on was rediscovered by D.Johnson and named as *lantern relation* by him [7],[11],[8]. For more results on lantern relation see [15].

The following relation was also known to Dehn and it is called the *chain relation.*

Let $\Sigma_{1,2}$ be a torus with two boundary components. If c_1 and c_2 are the boundary curves of $\Sigma_{1,2}$ and α_1, α_2, and β are the simple closed curves as shown in Figure 4, then we have

$$(t_{\alpha_1} t_\beta t_{\alpha_2})^4 = t_{c_1} t_{c_2}.$$

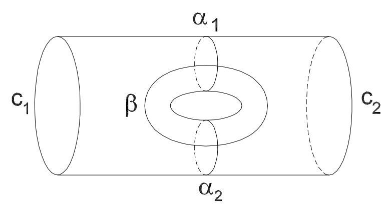


FIGURE 4. The Chain Relation

If c_1 bounds a disk then the chain relation becomes

$$(t_\alpha t_\beta t_\alpha)^4 = t_c,$$

where $\alpha = \alpha_1 = \alpha_2$ and $c = c_2$ is the only boundary curve. This is a special case of the chain relation and can be rewritten as

$$(t_\alpha t_\beta)^6 = t_c$$

using the braid relation $t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta$ twice, Figure 5.

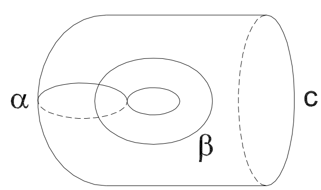


FIGURE 5. Special Case of the Chain Relation

One can also show that $(t_\beta t_\alpha)^6 = t_c$, using the braid relation.

If we call an ordered set of n simple closed curves c_n, where $c_i \perp c_{i+1}$ and $c_i \cap c_j = \emptyset$ for $|i - j| > 1$, *a chain of length* n [20], then the chain relation defined above is realized by a chain of length 3 and the special case of that relation shown in Figure 5 is realized by a chain of length 2.

A simple and explicit presentation for M_g, $g \geq 3$, was not given until 1983 [18]. Wajnryb gave the following presentation following the ideas of Hatcher-Thurston and Harer.

Theorem 1.3.9. *The mapping class group M_g of a 2-dimensional, closed, compact, oriented surface Σ_g of genus $g \geq 3$ admits a presentation with generators*

$$t_{d_2}, t_{c_1}, t_{c_2}, \ldots, t_{c_{2g}},$$

where the cycles $d_2, c_1, c_2, \ldots, c_{2g}$ are as shown in Figure 6, and with the following defining relations:

(*A*) *t_{d_2} and t_{c_4} are braided and t_{d_2} commutes with t_{c_i} for $i \neq 4$. t_{c_i} commutes with t_{c_j} if $|i-j| > 1$, and t_{c_i} and t_{c_j} are braided if $|i-j| = 1$, $1 \leq i, j \leq 2g$, Figure 6.*

(*B*) *$t_{c_1}, t_{c_2}, t_{c_3}, t_{d_2}$ and $t_{\widehat{d}_2}$ satisfy the chain relation, i.e., $(t_{c_1} t_{c_2} t_{c_3})^4 = t_{d_2} t_{\widehat{d}_2}$, where $\widehat{d}_2 = t_{c_4} t_{c_3} t_{c_2} t_{c_1}^2 t_{c_2} t_{c_3} t_{c_4} (d_2)$, Figure 6,7.*

(*C*) *$t_{c_1}, t_{c_3}, t_{c_5}, t_{d_3}, t_{d_2}, t_x$, and $t_{d_2 x}$ satisfy the lantern relation, i.e.,*

$$t_{c_1} t_{c_3} t_{c_5} t_{d_3} = t_{d_2} t_x t_{d_2 x},$$

where $x = t_{r_1} t_{r_2} (d_2)$, $t_{r_1} = t_{c_2} t_{c_1} t_{c_3} t_{c_2}$, $t_{r_2} = t_{c_4} t_{c_3} t_{c_5} t_{c_4}$, $d_3 = t_{c_6} t_{c_5} t_{c_4} t_{c_3} t_{c_2} t_a (b)$, $a = (t_{c_4} t_{c_3} t_{c_5} t_{c_4} t_{c_6} t_{c_5})^{-1} (d_2)$, and $b = (t_{c_4} t_{c_3} t_{c_2} t_{c_1})^{-1} (d_2)$, Figure 7.

(*D*) *$t_{c_1}, t_{c_2}, \ldots, t_{c_{2g}}$ and t_{d_g} satisfy the relation*

$$[t_{c_{2g}} t_{c_{2g-1}} \cdots t_{c_2} t_{c_1}^2 t_{c_2} \cdots t_{c_{2g-1}} t_{c_{2g}}, t_{d_g}] = 1,$$

where $d_g = t_{u_{g-1}} t_{u_{g-2}} \cdots t_{u_1} (c_1)$, $t_{u_i} = \left(t_{c_{2i-1}} t_{c_{2i}} t_{c_{2i+1}} t_{c_{2i+2}}\right)^{-1} t_{v_i} t_{c_{2i+2}} t_{c_{2i+1}} t_{c_{2i}}$ for $1 \leq i \leq g-1$, $v_1 = \widehat{d}_2$, $v_i = t_{r_{i-1}}^{-1} t_{r_i}^{-1} (v_{i-1})$ for $2 \leq i \leq g-1$, $t_{r_i} = t_{c_{2i}} t_{c_{2i+1}} t_{c_{2i-1}} t_{c_{2i}}$ for $1 \leq i \leq g-1$, Figure 7.

Relation (D) is related to the so called *hyperelliptic involution* $i : \Sigma_g \to \Sigma_g$ of Σ_g. It is the element of order 2, geometrically represented as the 180° rotation about the horizontal axis as shown in Figure 7. It fixes the unoriented cycles $c_1, c_2, \ldots, c_{2g}, d_g = c_{2g+1}$ and acts as $-Id$ on the homology. It can be shown that i can be expressed as $i = t_{c_{2g+1}} t_{c_{2g}} t_{c_{2g-1}} \cdots t_{c_2} t_{c_1}^2 t_{c_2} \cdots t_{c_{2g-1}} t_{c_{2g}} t_{c_{2g+1}}$.

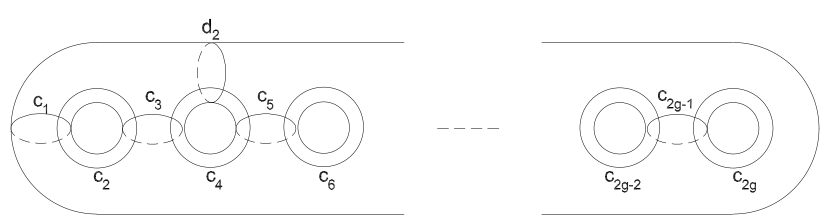


FIGURE 6. The Generators

From the expression for i it is clear that i commutes with t_{d_g} if and only if (D) holds.

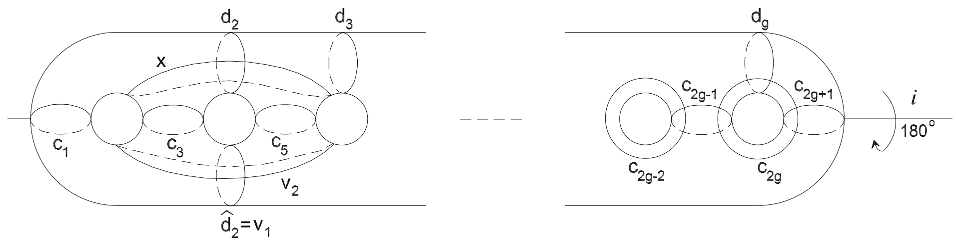


FIGURE 7. The Relations and the Hyperelliptic Involution

1.4. **Involutions.** The hyperelliptic involution $i : \Sigma_g \to \Sigma_g$ that is depicted in Figure 7 is probably the most studied and the best understood order 2 element in the mapping class group. The so-called *hyperelliptic mapping class group* H_g, is the infinite subgroup of M_g consisting of the elements that commute with i. All of the generators of M_g commute with i for $g = 1, 2$, therefore $M_g = H_g$, for $g = 1, 2$.

For higher genus, H_g is generated by t_{c_i}, $i = 1, \ldots, 2g+1$, where c_i are the cycles that are shown in Figure 7 . The relations are

$$\begin{aligned} t_{c_i} t_{c_j} &= t_{c_j} t_{c_i} \text{ if } |i-j| > 1, \\ t_{c_i} t_{c_{i+1}} t_{c_i} &= t_{c_{i+1}} t_{c_i} t_{c_{i+1}}, \\ \left(t_{c_{2g+1}} t_{c_{2g}} \cdots t_{c_2} t_{c_1}^2 t_{c_2} \cdots t_{c_{2g}} t_{c_{2g+1}}\right)^2 &= 1, \\ \text{and } \left(t_{c_1} t_{c_2} \cdots t_{c_{2g}} t_{c_{2g+1}}\right)^{2g+2} &= 1. \end{aligned}$$

In particular this gives a presentation for M_2 because $M_2 = H_2$.

The orbit space of the involution i is the sphere with 6 marked points, denote it by $\Sigma_{0,6}$. It defines a $2-$ fold branched covering $p : \Sigma_2 \to \Sigma_{0,6}$, branched at 6 points as shown in Figure 8. The cycles $c_i, i = 1, \ldots, 5$ in Σ_2 project to the segments $p(c_i)$ in $\Sigma_{0,6}$ that connect the marked points q_i and q_{i+1}. The isotopy classes of the half twists about $p(c_i)$, which we denote by w_i for $i = 1, \ldots, 5$, generate the mapping class group $M\left(S^2, 6\right)$ of $\Sigma_{0,6}$ (The marked points are fixed setwise). There is a surjective homomorphism

$$\psi : H_2 \to M\left(S^2, 6\right)$$

sending t_{c_i} to $w_i, i = 1, \ldots, 5$, and with $\ker \psi =< i >$. For a proof see [1].

There is another involution in the mapping class group which is also geometric. We will denote it by s. It is described as 180^o rotation about the vertical axis as shown in Figure 9 . We will begin with finding an explicit positive Dehn twist expression for s in M_2. The following paragraph explains the idea, which generalizes to M_g with some extra work. See [14], [9] for details.

Since s commutes with i in M_2, it descends to a map $\widetilde{s} : \Sigma_{0,6} \to \Sigma_{0,6}$. We will obtain an expression for $\widetilde{s}$ in terms of the generators w_i and then lift that expression to $H_2 = M_2$ via the surjection ψ.

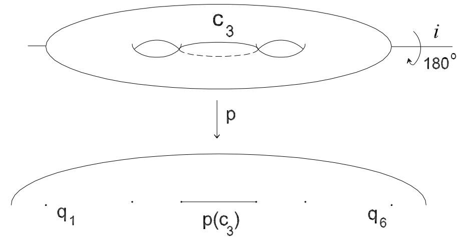


FIGURE 8. Two Fold Branched Cover Defined by i

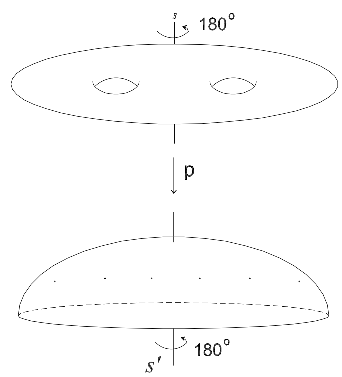


FIGURE 9. The Involution s

$\widetilde{s}$ is counterclokwise rotation through 180^o on $\Sigma_{0,6}$, about the axis through the center and the south pole of the sphere, fixing the marked points setwise. It fixes the south pole, therefore forgetting that point we can isotope $\widetilde{s}$ to a map s' of a disc including the 6 marked points, Figure 9. Being an element of $M\left(D,6\right)=B_6$, s' can be realized as a braid. Figure 10 (a) and (b) show two different ways of sketching that braid. The first one has the expression

$$\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1$$

and the second one has the expression

$$\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3\sigma_5\sigma_4\sigma_5.$$

Since Figure 10 (a) and (b) are representing isotopic braids, they correspond to the same element in $M\left(D,6\right)$. Therefore

$$s'^2=\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3\sigma_5\sigma_4\sigma_5\underline{\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1}\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1$$

which is equal to

$$\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\underline{\sigma_1}\sigma_5\sigma_4\sigma_3\underline{\sigma_2\sigma_1}\sigma_5\sigma_4\underline{\sigma_3\sigma_2\sigma_1}\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1.$$

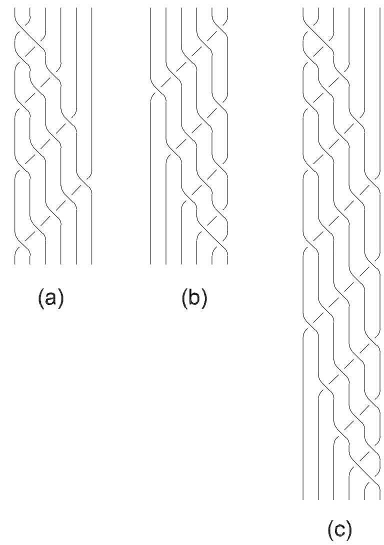


FIGURE 10. Braids Representing s' and s'^2

Thus

$$s'^2 = (\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1)^6$$

using commutativity relations only. It is not difficult to see that this is isotopic to a right handed twist about the boundary of the disk D, Figure 10 (c). Therefore we have

$$\begin{aligned} \widetilde{s}^2 &= (w_5w_4w_3w_2w_1)^6 \\ &= 1 \end{aligned}$$

in $M\left(S^2, 6\right)$ and hence the lift h of $\widetilde{s}$ satisfies

$$h^2 = (t_{c_5}t_{c_4}t_{c_3}t_{c_2}t_{c_1})^6 ,$$

which is equal to 1 in M_2. One needs to check the action of s and

$$h = t_{c_1}t_{c_2}t_{c_1}t_{c_3}t_{c_2}t_{c_1}t_{c_4}t_{c_3}t_{c_2}t_{c_1}t_{c_5}t_{c_4}t_{c_3}t_{c_2}t_{c_1}$$

on the homology to see that the action of s is the same as the action of h, but not as that of $h \circ i$ [14]. Therefore $s = h$ and

$$s^2 = (t_{c_5}t_{c_4}t_{c_3}t_{c_2}t_{c_1})^6 .$$

Using the braid and commutativity relations it is not difficult to show that

$$s^2 = (t_{c_1}t_{c_2}t_{c_3}t_{c_4}t_{c_5})^6$$

as well.

Another positive Dehn twist expression for s, which is of particular interest from topological point of view [16], is

$$s = t_{b_0}t_{b_1}t_{b_2}t_c,$$

where b_0, b_1, b_2, and c are the cycles shown in Figure 11.

FIGURE 11. Another Dehn Twist Expression for s

Distinguishing the cycle notation from the Dehn twist notation, it is easy to see that

$$\begin{aligned} b_0 &= t_{c_1}t_{c_2}t_{c_3}t_{c_4}(c_5), \\ b_1 &= t_{c_1}t_{c_1}t_{c_2}t_{c_3}(c_4), \\ b_2 &= t_{c_2}t_{c_1}t_{c_1}t_{c_2}(c_3). \end{aligned}$$

Applying Lemma 1.3.5 we get

$$\begin{aligned} t_{b_0} &= t_{c_1}t_{c_2}t_{c_3}t_{c_4}t_{c_5}\left(t_{c_1}t_{c_2}t_{c_3}t_{c_4}\right)^{-1}, \\ t_{b_1} &= t_{c_1}t_{c_1}t_{c_2}t_{c_3}t_{c_4}\left(t_{c_1}t_{c_1}t_{c_2}t_{c_3}\right)^{-1}, \\ t_{b_2} &= t_{c_2}t_{c_1}t_{c_1}t_{c_2}t_{c_3}\left(t_{c_2}t_{c_1}t_{c_1}t_{c_2}\right)^{-1}. \end{aligned}$$

Therefore the product $t_{b_0}t_{b_1}t_{b_2}$ becomes

$$t_{c_1}t_{c_2}t_{c_3}t_{c_4}t_{c_5}\left(t_{c_1}t_{c_2}t_{c_3}t_{c_4}\right)^{-1}t_{c_1}t_{c_1}t_{c_2}t_{c_3}t_{c_4}\left(t_{c_1}t_{c_1}t_{c_2}t_{c_3}\right)^{-1}t_{c_2}t_{c_1}t_{c_1}t_{c_2}t_{c_3}\left(t_{c_2}t_{c_1}t_{c_1}t_{c_2}\right)^{-1}.$$

We also have $t_c = \left(t_{c_1}t_{c_2}t_{c_1}\right)^4 = \left(t_{c_1}t_{c_2}\right)^6 = \left(t_{c_2}t_{c_1}\right)^6$ using the braid relation and the special case of the chain relation on the subsurface Σ_1^1.

Next, we will simplify the expression for the product $t_{b_0}t_{b_1}t_{b_2}$. We will use just the indices representing the twists for brevity. For example 1 will mean t_{c_1}, 2 will mean t_{c_2}, etc. The change in each line occurs within the underlined portion of the entire expression and the result of that change is not underlined in the next line. We only use commutativity and braid relations.

$$
\begin{aligned}
t_{b_0}t_{b_1}t_{b_2} &= 12345\,(\underline{1}234)^{-1}\,\underline{1}1234\,(1123)^{-1}\,21123\,(2112)^{-1}\\
&= 12345\underline{(234)^{-1}\,1234}\,(1123)^{-1}\,21123\,(2112)^{-1}\\
&= 123451234\,(123)^{-1}\,(\underline{11}23)^{-1}\,21123\,(2112)^{-1}\\
&= 123451234\,(123)^{-1}\,(23)^{-1}\,\underline{(11)^{-1}}21123\,(2112)^{-1}\\
&= 123451234\,(123)^{-1}\,(\underline{2}3)^{-1}\,21123\,(11)^{-1}\,(2112)^{-1}\\
&= 123451234\,(123)^{-1}\,\underline{(3)^{-1}}1123\,(11)^{-1}\,(2112)^{-1}\\
&= 123451234\,(\underline{1}23)^{-1}\,1123\,(2)^{-1}\,(11)^{-1}\,(2112)^{-1}\\
&= 123451234\underline{(23)^{-1}}123\,(2)^{-1}\,(11)^{-1}\,(2112)^{-1}\\
&= 123451234123\,(12)^{-1}\,(2)^{-1}\,(11)^{-1}\,(2112)^{-1}\\
&= 123451234123\underline{1211^{-1}\,(12)^{-1}}\,(12)^{-1}\,(2)^{-1}\,(11)^{-1}\,(2112)^{-1}\\
&= 1234512341231211^{-1}2^{-1}1^{-1}2^{-1}1^{-1}2^{-1}1^{-1}\underline{1^{-1}2^{-1}1^{-1}}1^{-1}2^{-1}\\
&= 1234512341231211^{-1}2^{-1}1^{-1}2^{-1}1^{-1}2^{-1}1^{-1}2^{-1}1^{-1}2^{-1}1^{-1}2^{-1}\\
&= 123451234123121\,\left(1^{-1}2^{-1}\right)^{6}\\
&= 123451234123121\,(21)^{-6}\,.
\end{aligned}
$$

Now using the fact that $t_c = (21)^6$ we get

$$
\begin{aligned}
t_{b_0}t_{b_1}t_{b_2}t_c &= 123451234123121\,(21)^{-6}\,(21)^{6}\,,\\
t_{b_0}t_{b_1}t_{b_2}t_c &= 123451234123121,
\end{aligned}
$$

i.e.,

$$s = t_{b_0}t_{b_1}t_{b_2}t_c = t_{c_1}t_{c_2}t_{c_3}t_{c_4}t_{c_5}t_{c_1}t_{c_2}t_{c_3}t_{c_4}t_{c_1}t_{c_2}t_{c_3}t_{c_1}t_{c_2}t_{c_1}.$$

To see that this is the same expression that we obtained earlier:

$$
\begin{aligned}
s &= t_{c_1}t_{c_2}t_{c_3}t_{c_4}t_{c_5}\underline{t_{c_1}}t_{c_2}t_{c_3}t_{c_4}t_{c_1}t_{c_2}t_{c_3}t_{c_1}t_{c_2}t_{c_1}\\
&= t_{c_1}t_{c_2}t_{c_1}t_{c_3}t_{c_4}t_{c_5}\underline{t_{c_2}}t_{c_3}t_{c_4}\underline{t_{c_1}}t_{c_2}t_{c_3}t_{c_1}t_{c_2}t_{c_1}\\
&= t_{c_1}t_{c_2}t_{c_1}t_{c_3}t_{c_2}t_{c_1}t_{c_4}t_{c_5}\underline{t_{c_3}}t_{c_4}\underline{t_{c_2}}t_{c_3}\underline{t_{c_1}}t_{c_2}t_{c_1}\\
&= t_{c_1}t_{c_2}t_{c_1}t_{c_3}t_{c_2}t_{c_1}t_{c_4}t_{c_3}t_{c_2}t_{c_1}t_{c_5}t_{c_4}t_{c_3}t_{c_2}t_{c_1}.
\end{aligned}
$$

For an excellent treatment of this work for general case see [9].

2. Main Theorem

The hyperelliptic involution $i : f : \Sigma_h \to \Sigma_h$ and the involution $s : \Sigma_k \to \Sigma_k$ that were described in section 1.4 can be combined into an involution $\theta : \Sigma_{h+k} \to \Sigma_{h+k}$ that supports the action of both i and s on bounded subsurfaces of Σ_{h+k} as shown in Figure 12.

The main theorem gives a positive Dehn twist expression for θ. We will use the same notation for both the cycles and the Dehn twists about them throughout the section. The order of the product is from right to left.

Theorem 2.0.1. *The positive Dehn twist expression for the involution that is shown in Figure 12 is given by*

$$\theta = c_{2i+2}\cdots c_{2h}c_{2h+1}c_{2i}\cdots c_2c_1b_0c_{2h+1}c_{2h}\cdots c_{2i+2}c_1c_2\cdots c_{2i}b_1b_2\cdots b_{k-1}b_kc_{2i+1}.$$

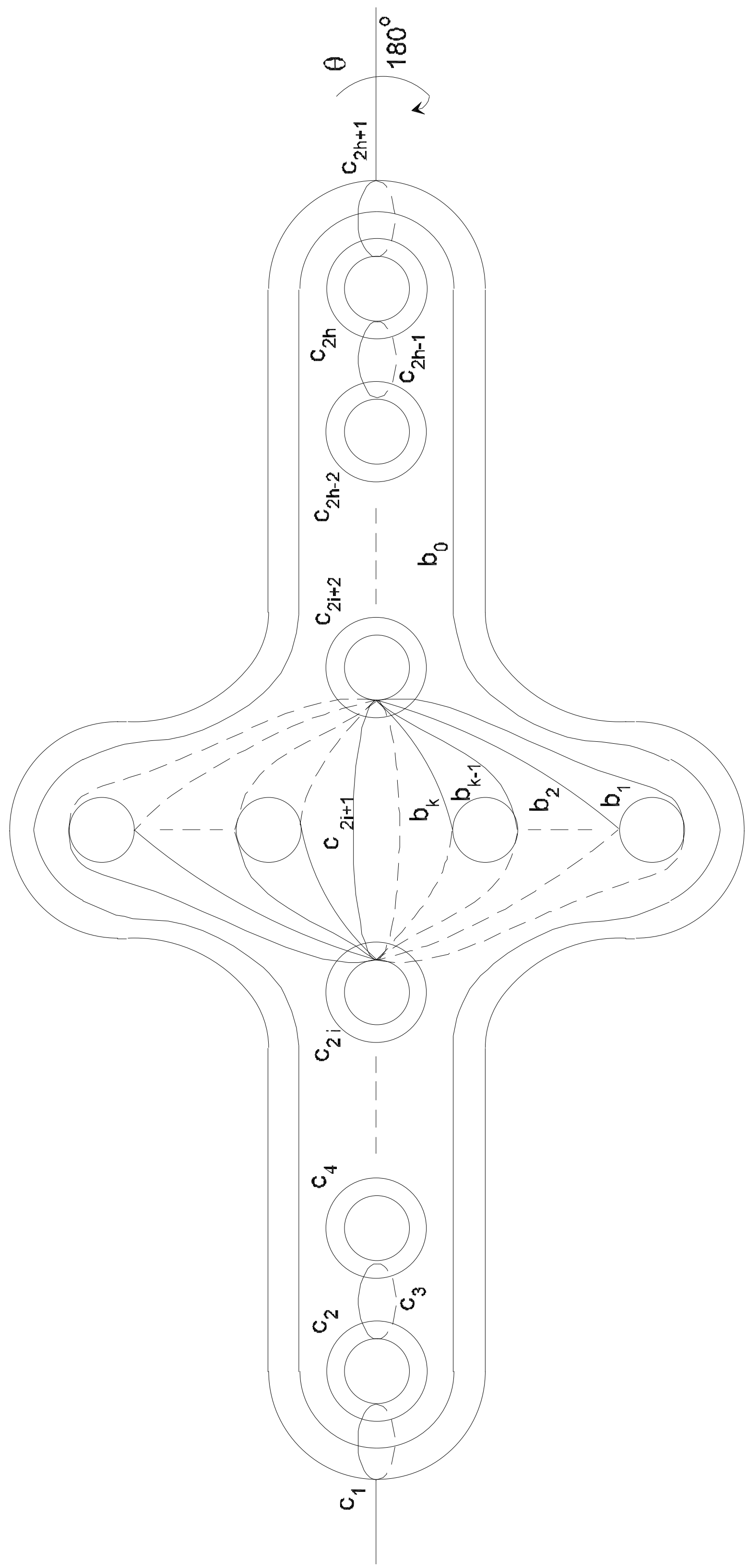


FIGURE 12. The Involution θ

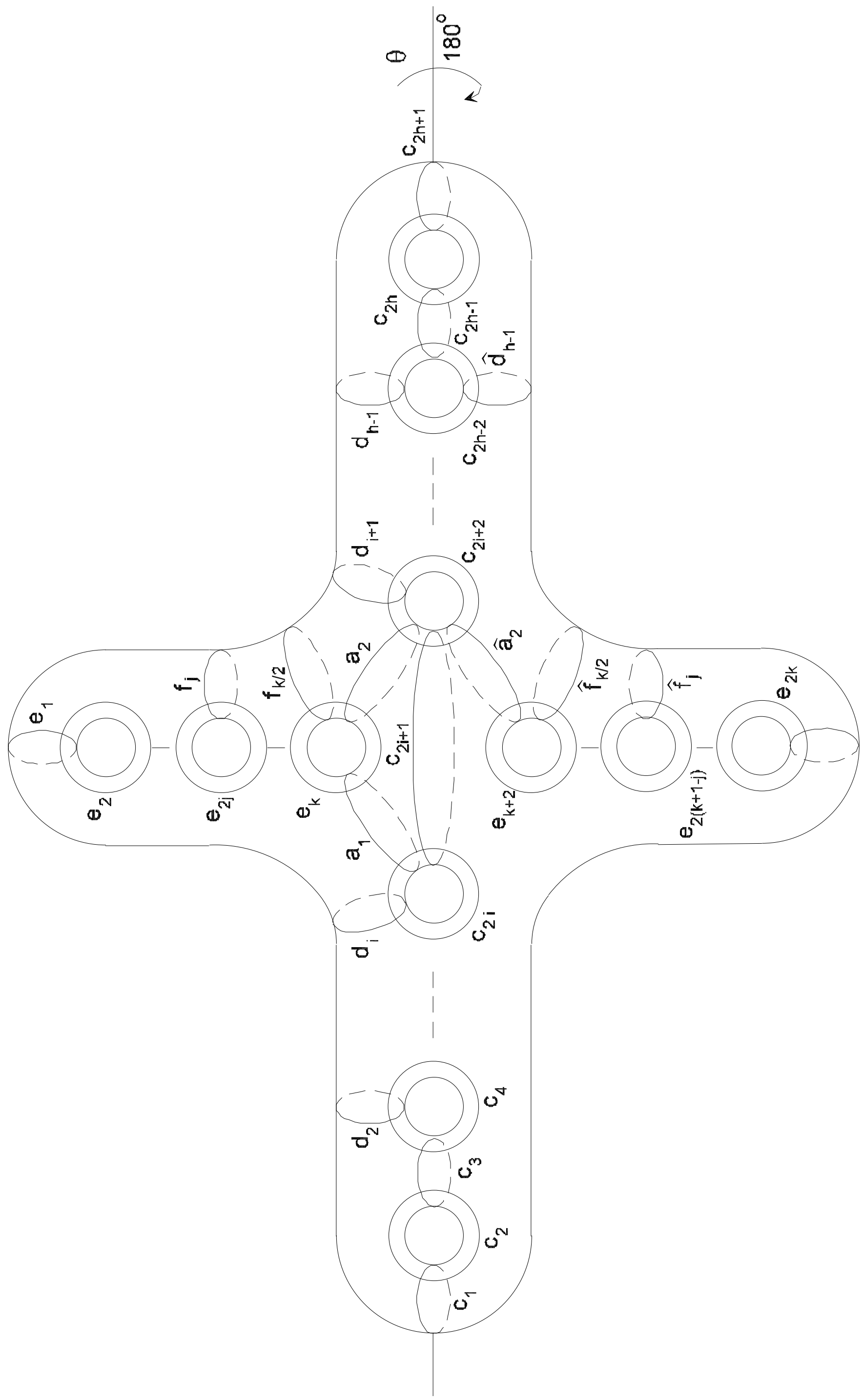


FIGURE 13. Base Cycles

Proof: Figure 13 shows a set of cycles that will constitute a *base* in the sense that, mapping of those cycles will ensure the mapping of the subsurfaces that they bound accordingly. Therefore, it suffices to check the images of those cycles under θ and see that they are mapped to the correct places. Furthermore, for cycle pairs that are symmetrical with respect to the vertical axis through the center of the figure, we will look at the mapping of those that are on the right hand side only. The mapping of their counterpart goes similarly due to symmetry. Finally, Figure 22 shows an example for the mapping of separating cycles.

First we will see the mapping of c_j for $j \neq 2i, 2i+1$ or $2i+2$. Figure 14 shows the mapping of such a cycle for $2i+3 \leq j \leq 2h+1$. Whether c_j goes around a hole or goes through two holes, only two twists are effective in its mapping: c_j and c_{j-1}.

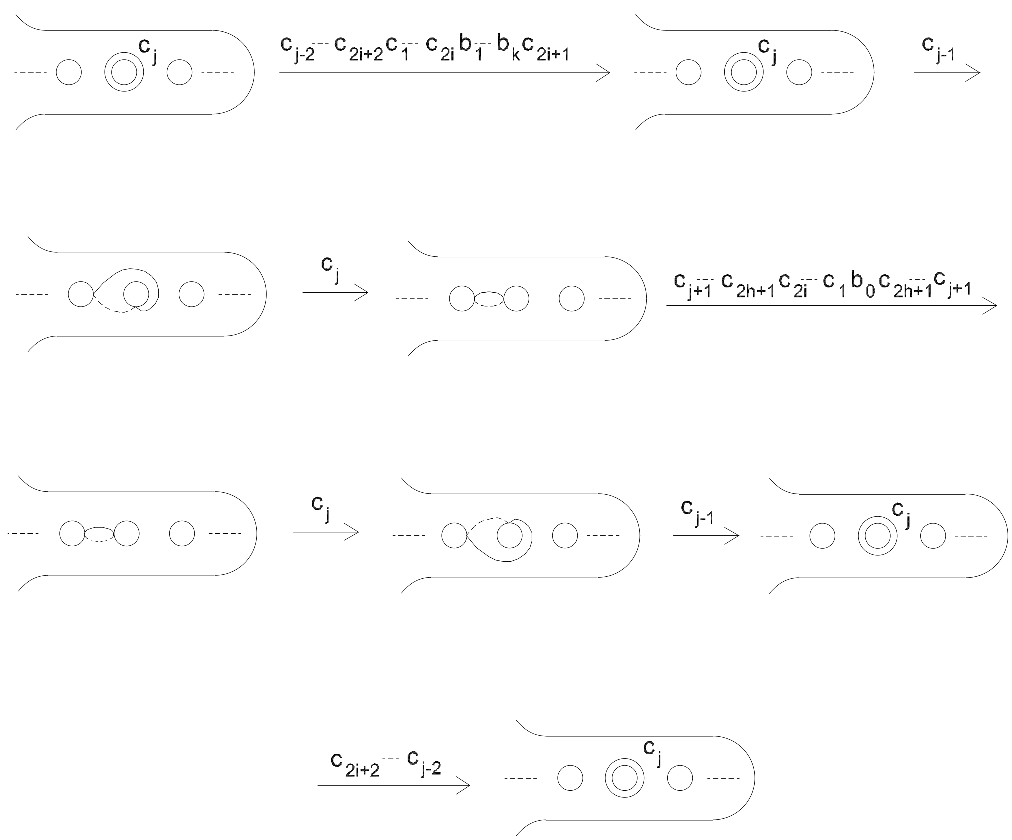


FIGURE 14. Mapping of c_j

Next, we will find the image of d_j for $2 \le j \le h-1$. Figure 15 shows the mapping of d_j for $i+1 \le j \le h-1$. As the figure indicates the twists along $b_j, j = 0, 1, \ldots, k$ do not take part in mapping of d_j. This is true for d_i and d_{i+1} as well.

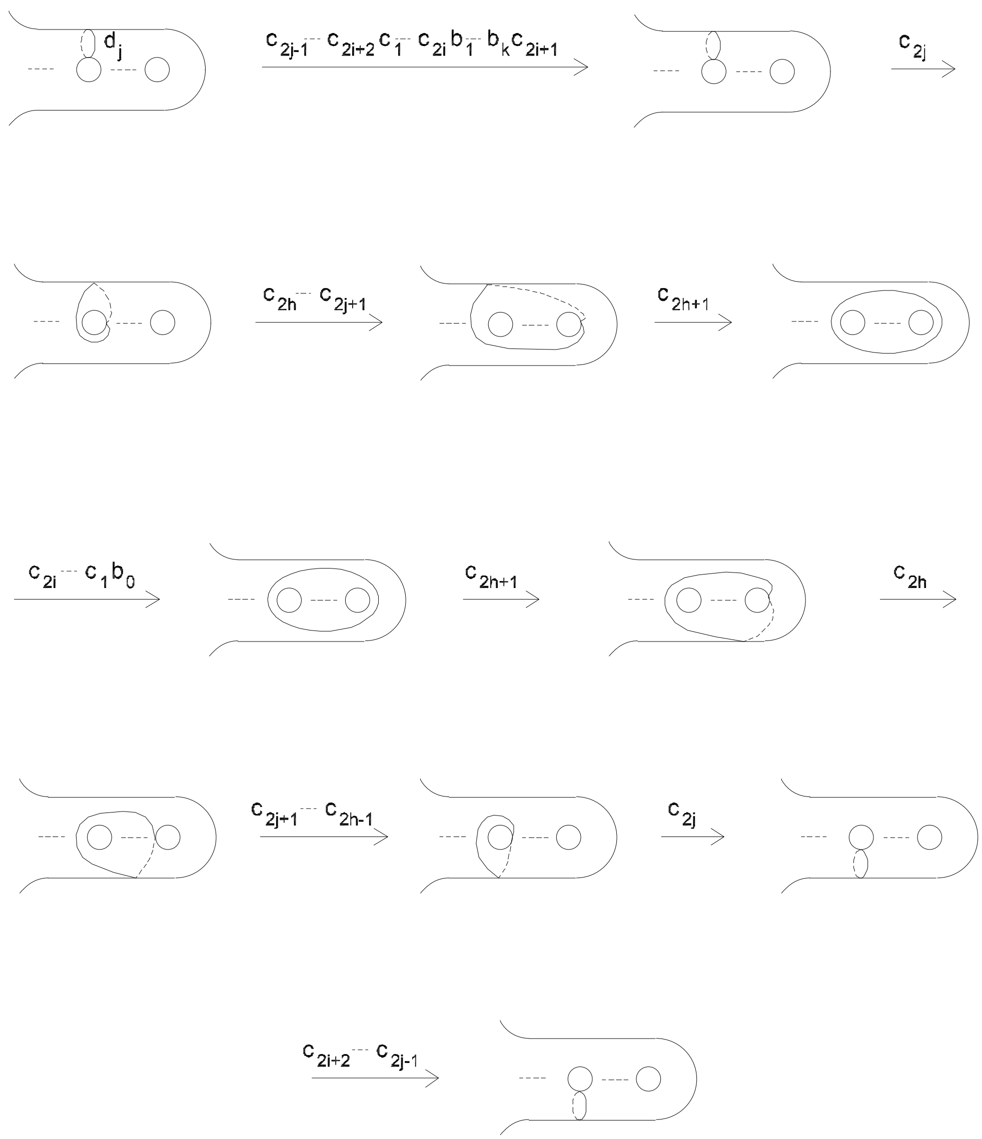
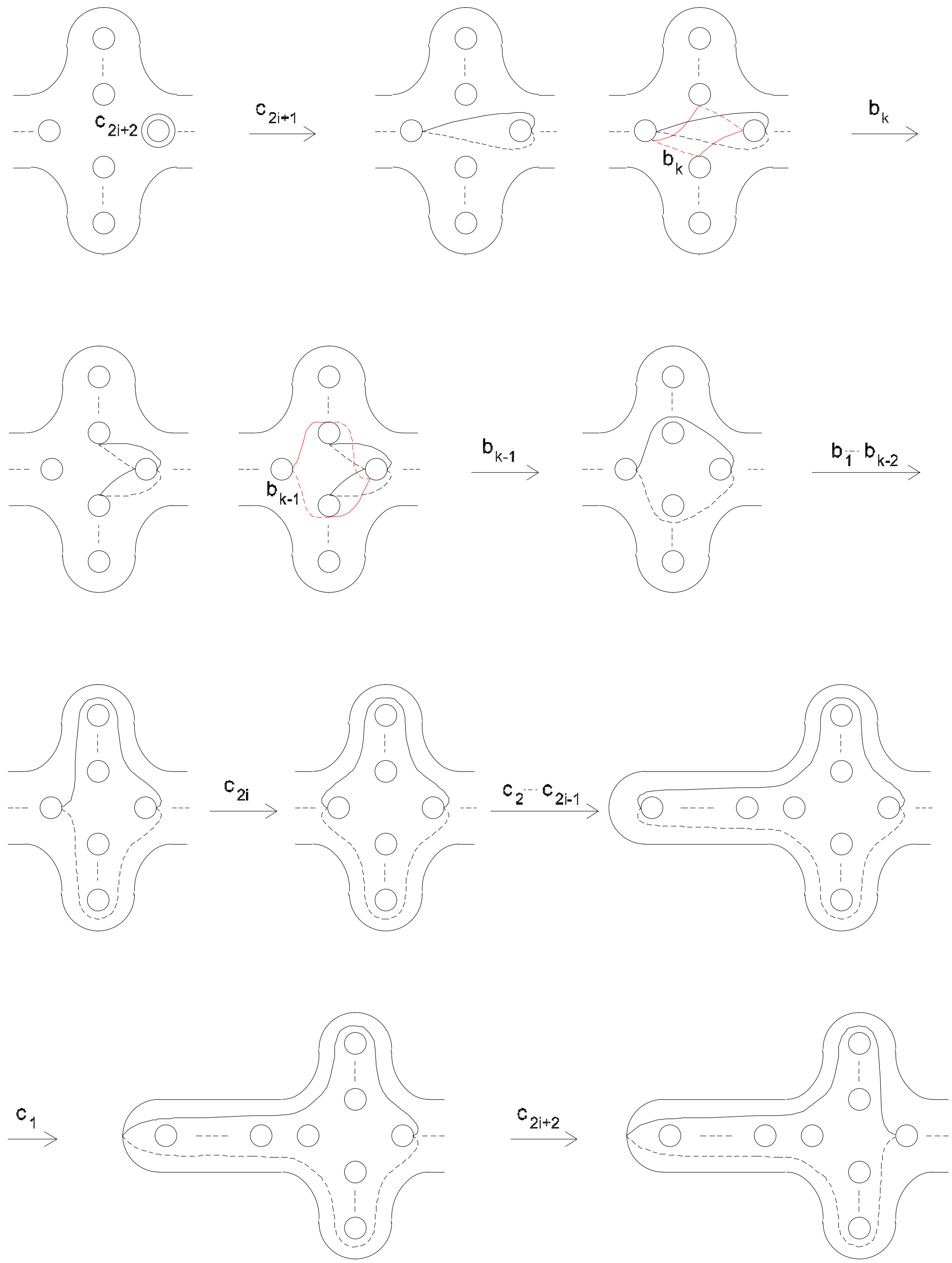


FIGURE 15. Mapping of d_j

Figure 16 shows how c_{2i+2} is mapped. The mapping of c_{2i} is similar, therefore we omit the proof.



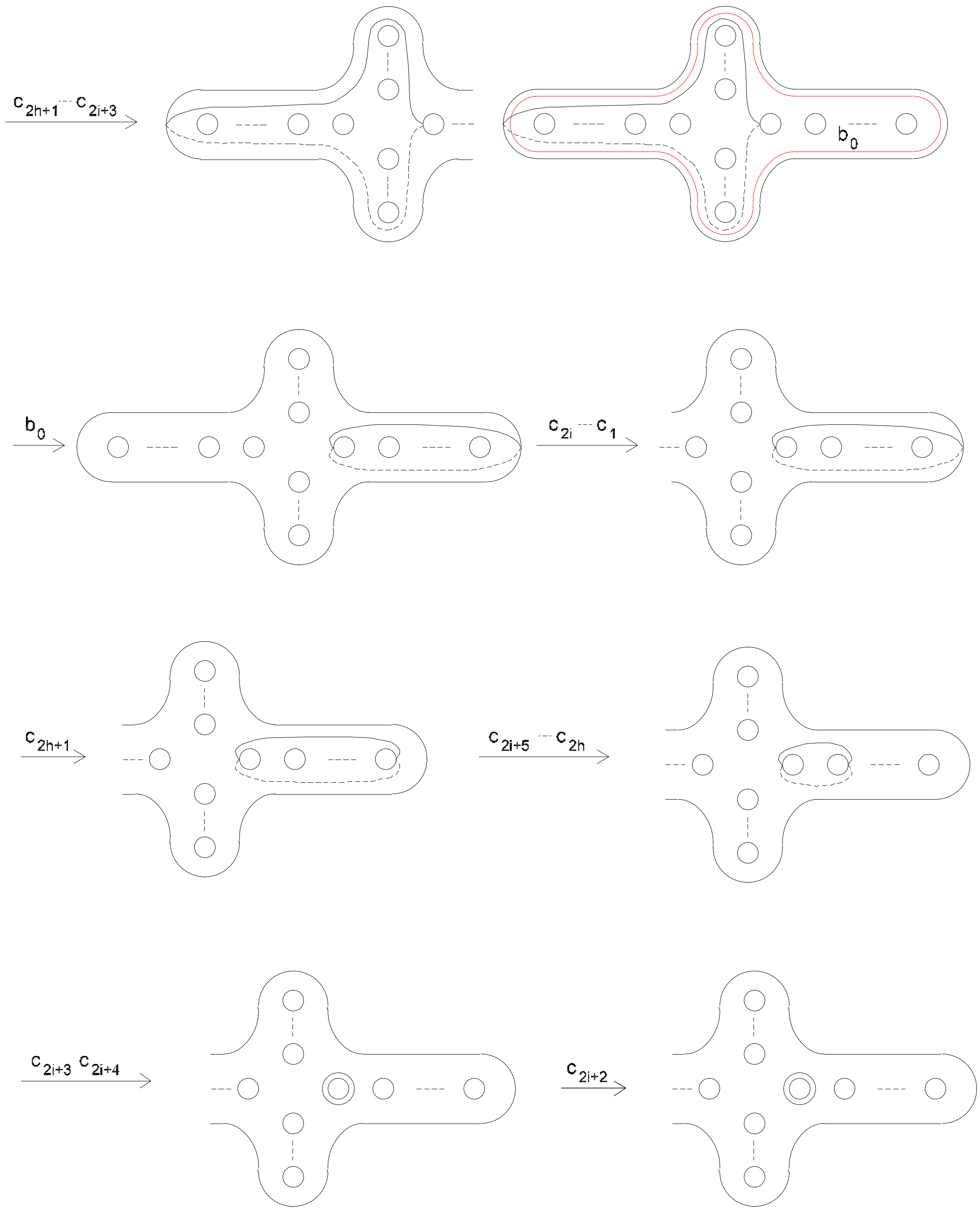


FIGURE 16. Mapping of c_{2i+2}

Figure 17 shows the mapping of e_1. b_0 is effective in the mapping of e_1 as the figure indicates and it does not take part in the mapping of $e_i, i = 2, \ldots, 2k$. The mapping of e_{2k+1} is the same as that of e_1, therefore its proof is omitted. The cycle to which b_0 is applied looks different in the third line but they are isotopic.

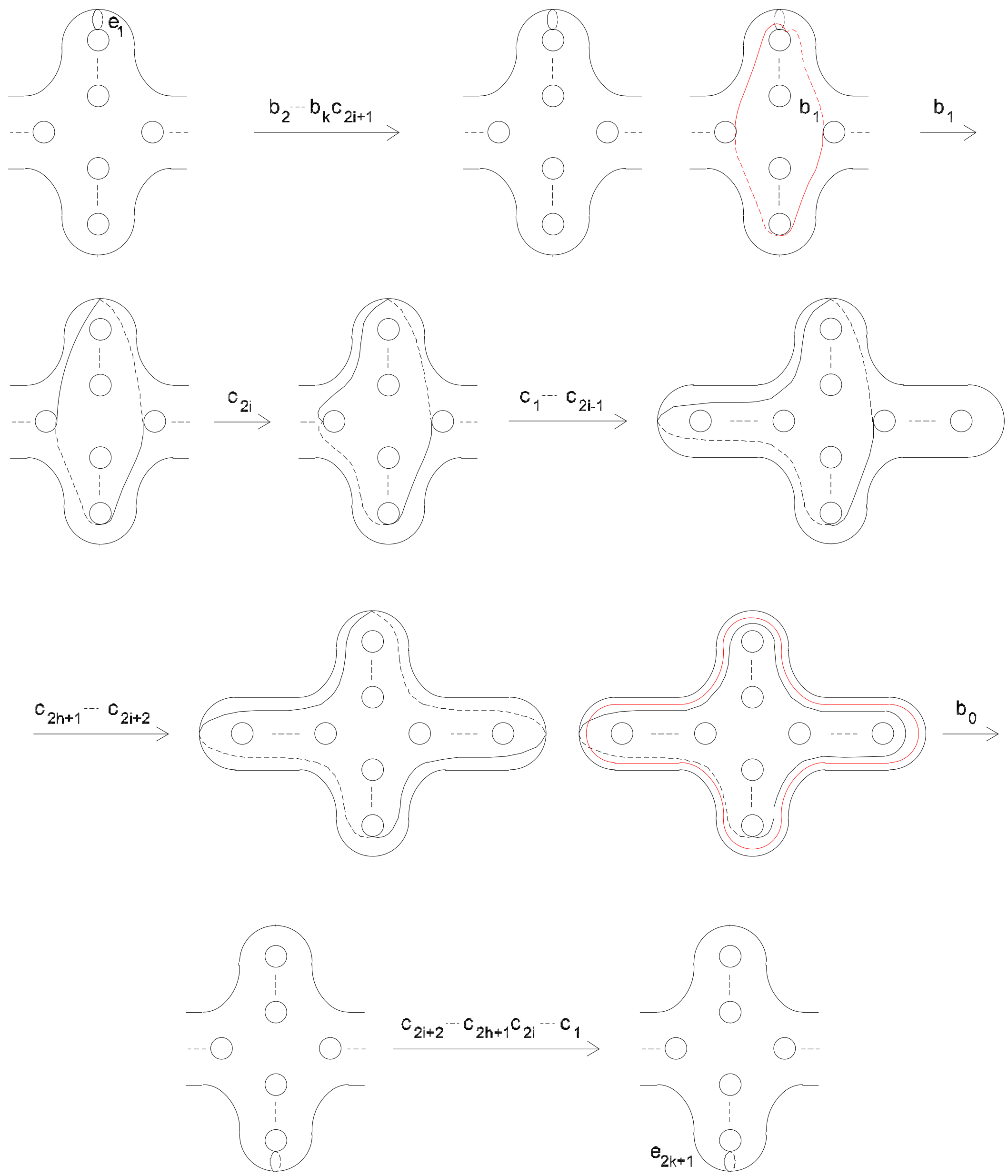


FIGURE 17. Mapping of e_1

The mapping of e_2 is shown in the first two lines of Figure 18. The mapping of e_j for $j-$ even is similar to that of e_2, including e_k. The mapping of $e_{2j-1}, j = 2, \ldots, k/2$ is shown in the last two lines of the same figure. The proof for the mapping of $e_j, j = k+2, \ldots, 2k$ is similar, therefore omitted.

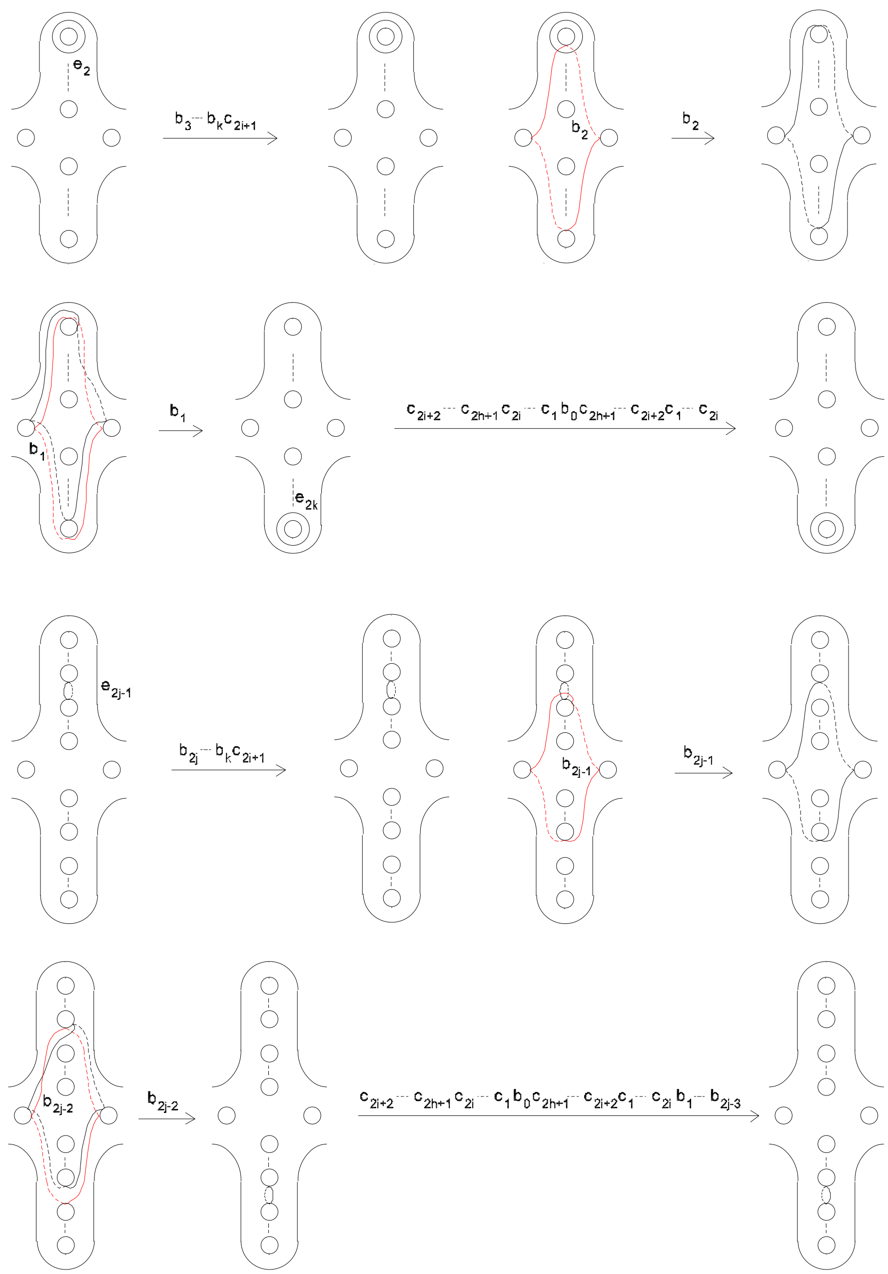


FIGURE 18. Mappings of e_2 and $e_{2j-1}, j = 2, \ldots, k/2$

Figure 19 shows the mapping of $f_{k/2}$ and the mapping of $f_j, j = 2, \ldots, k/2-1$ are the same. We omit the proof for $f_j, j = k/2+1, \ldots, k$ due to the same reason. The cycle to which the twist about b_0 is applied in the last line looks different in the previous figure but they are isotopic.

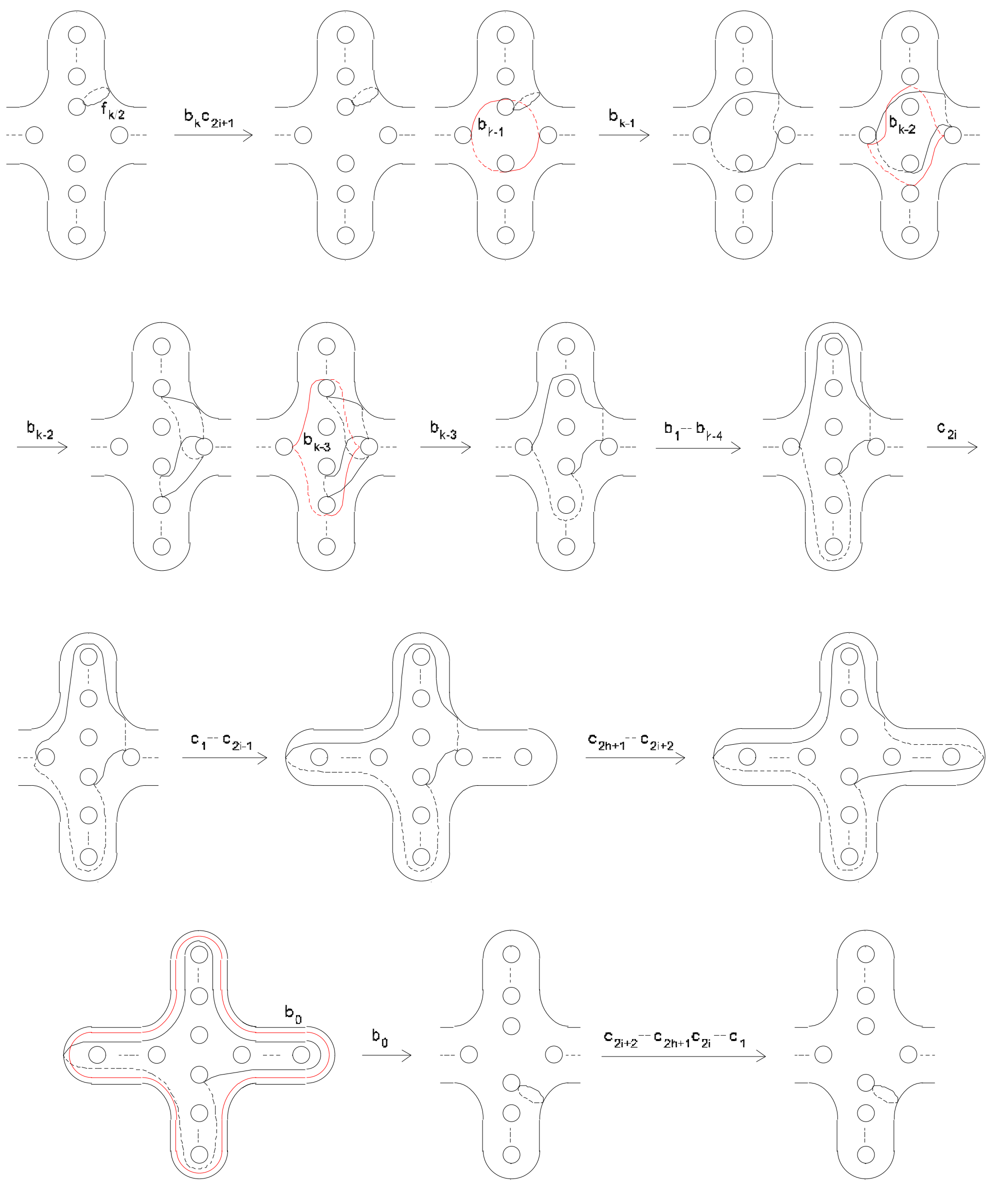


FIGURE 19. Mapping of $f_{k/2}$

The mapping of a_2 is shown in Figure 20. The proof for the mapping of its mirror image is omitted because of the symmetry.

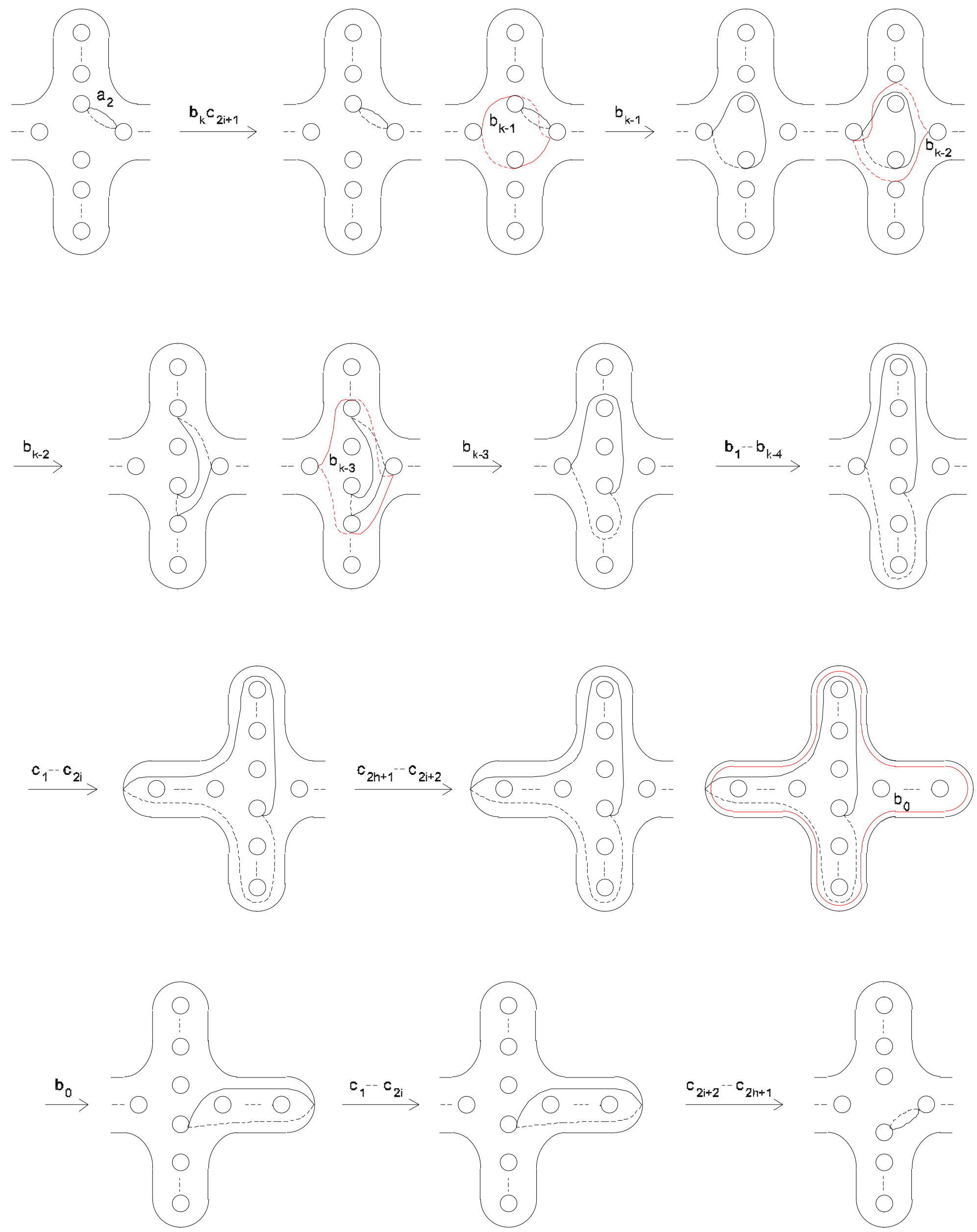
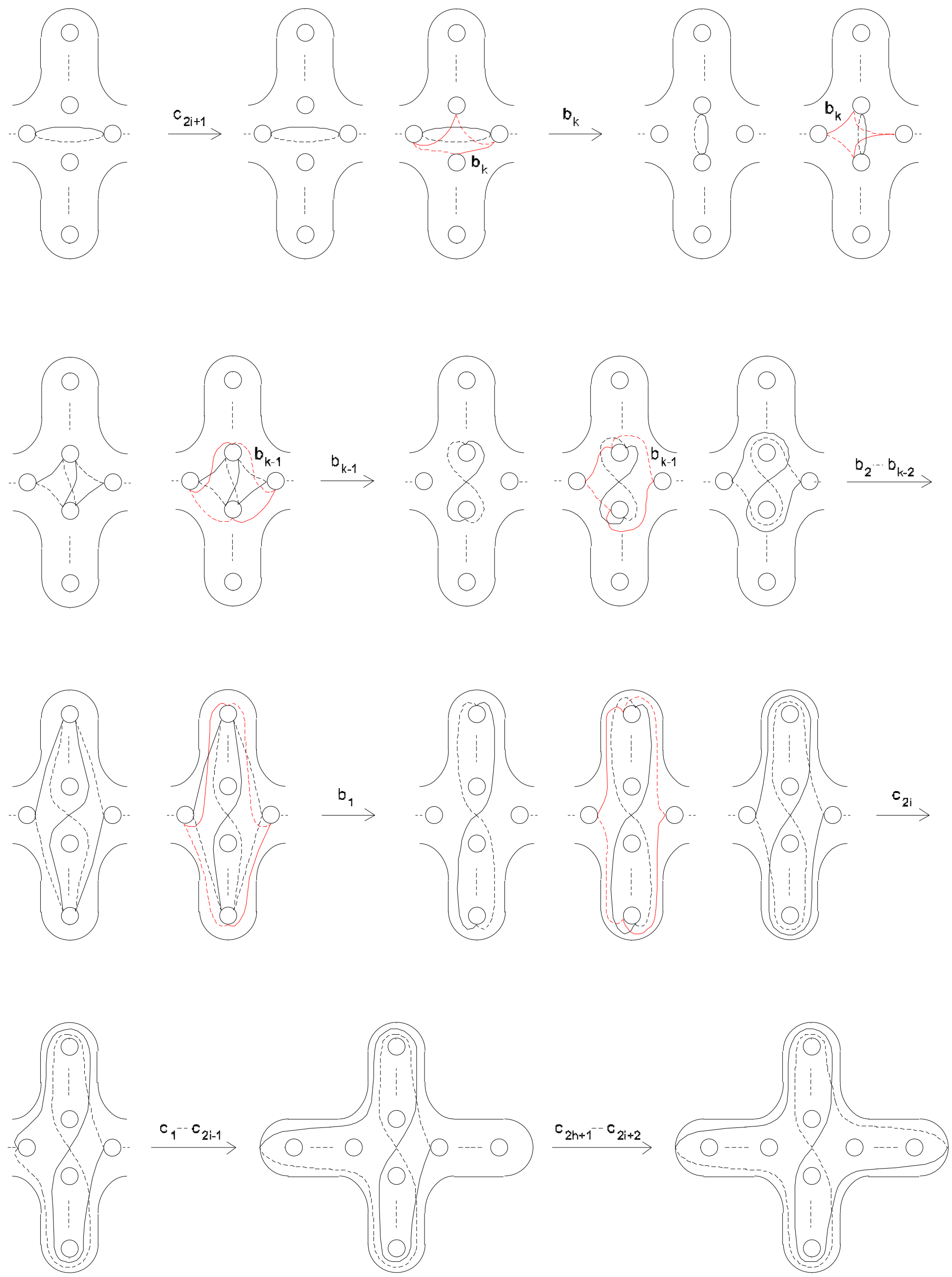


FIGURE 20. Mapping of a_2

Figure 21 shows the mapping of c_{2i+1}. Since $c_{2i+1} \cap b_k = 2$, we have $t_{b_k}(c_{2i+1}) = b_k^2\ c_{2i+1}$ by Lemma 1.3.3. Also since $t_{b_k}(c_{2i+1}) \cap b_{k-1} = 2$ we have $t_{b_{k-1}}(t_{b_k}(c_{2i+1})) = b_{k-1}^2\ (b_k^2\ c_{2i+1})$. Lemma 1.3.3 applies to all $b_j, j = 0, \ldots, k$ because each of them intersects the curve they are applied to twice.



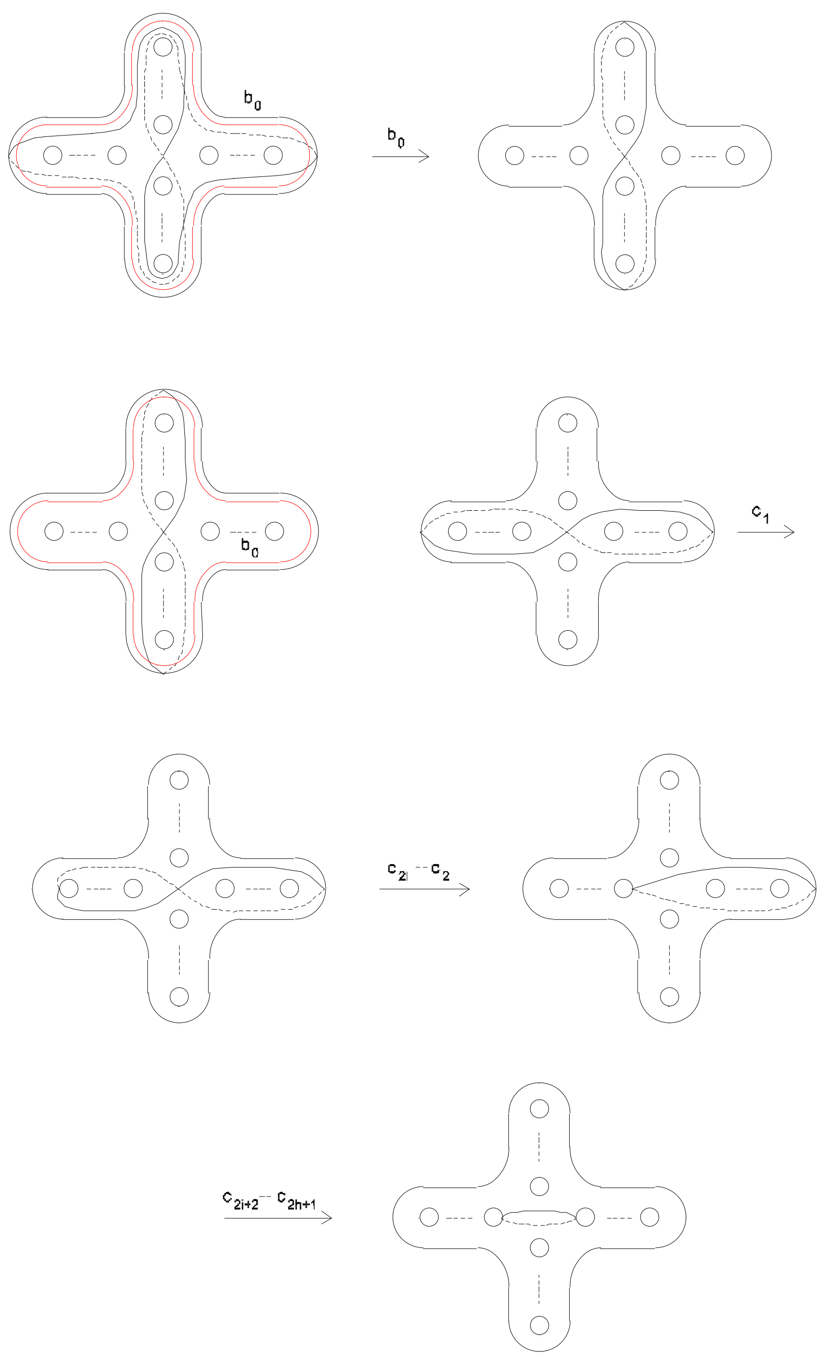
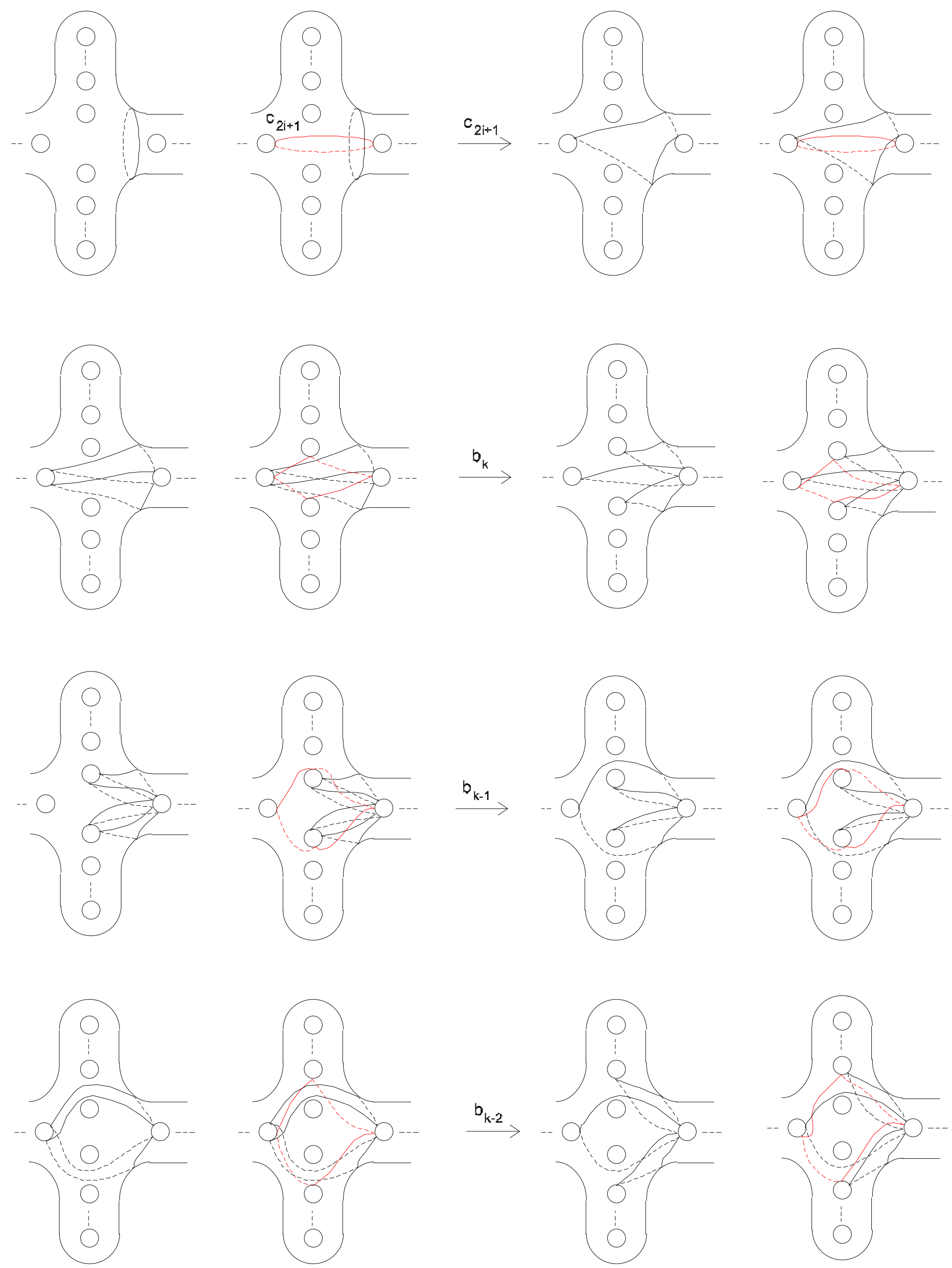
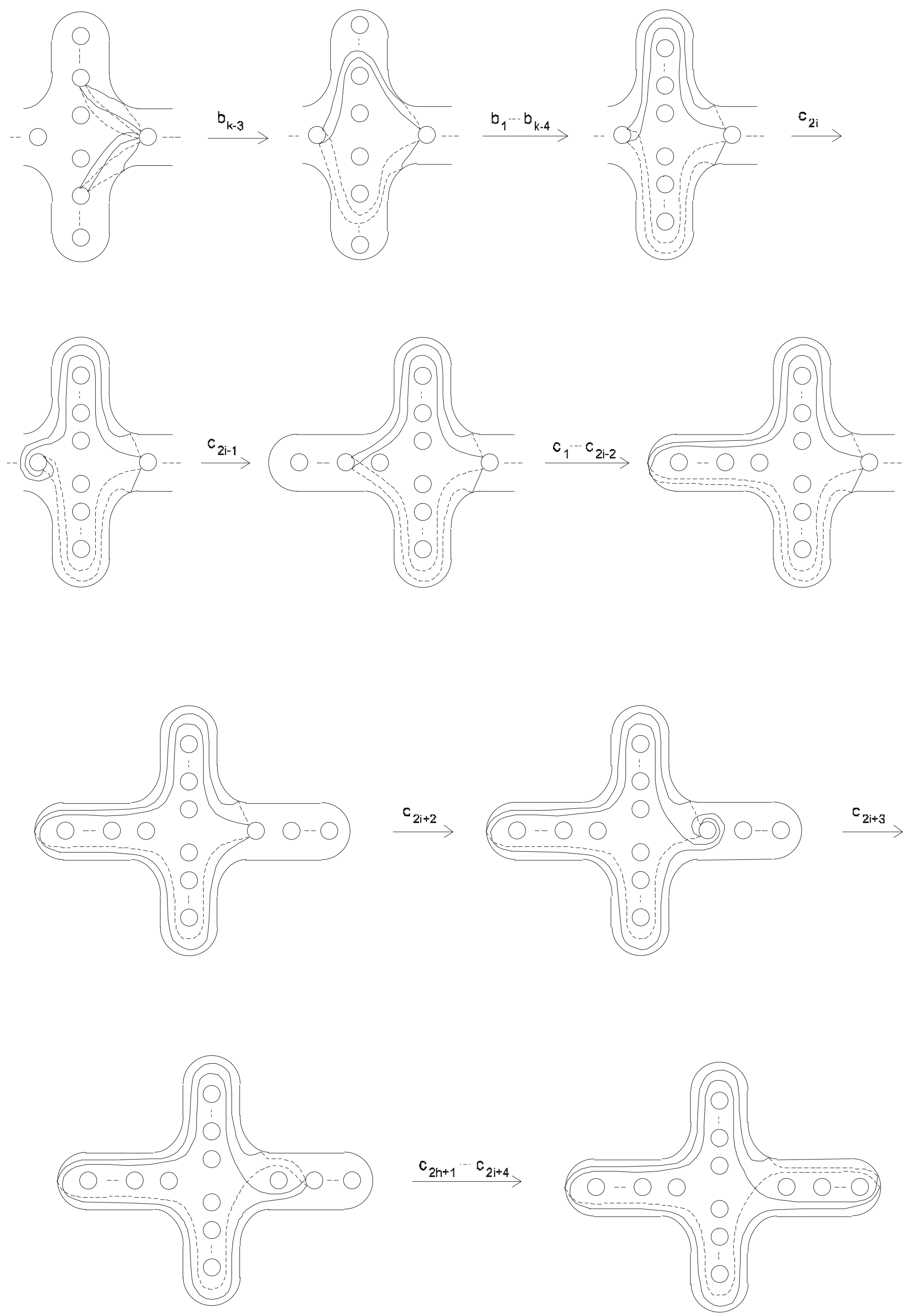


FIGURE 21. Mapping of c_{2i+1}

Figure 22 shows the mapping of a separating cycle.





b_{k-3}
$b_1 \cdots b_{k-4}$
c_{2i}
c_{2i-1}
$c_1 \cdots c_{2i-2}$
c_{2i+2}
c_{2i+3}
$c_{2h+1} \cdots c_{2i+4}$

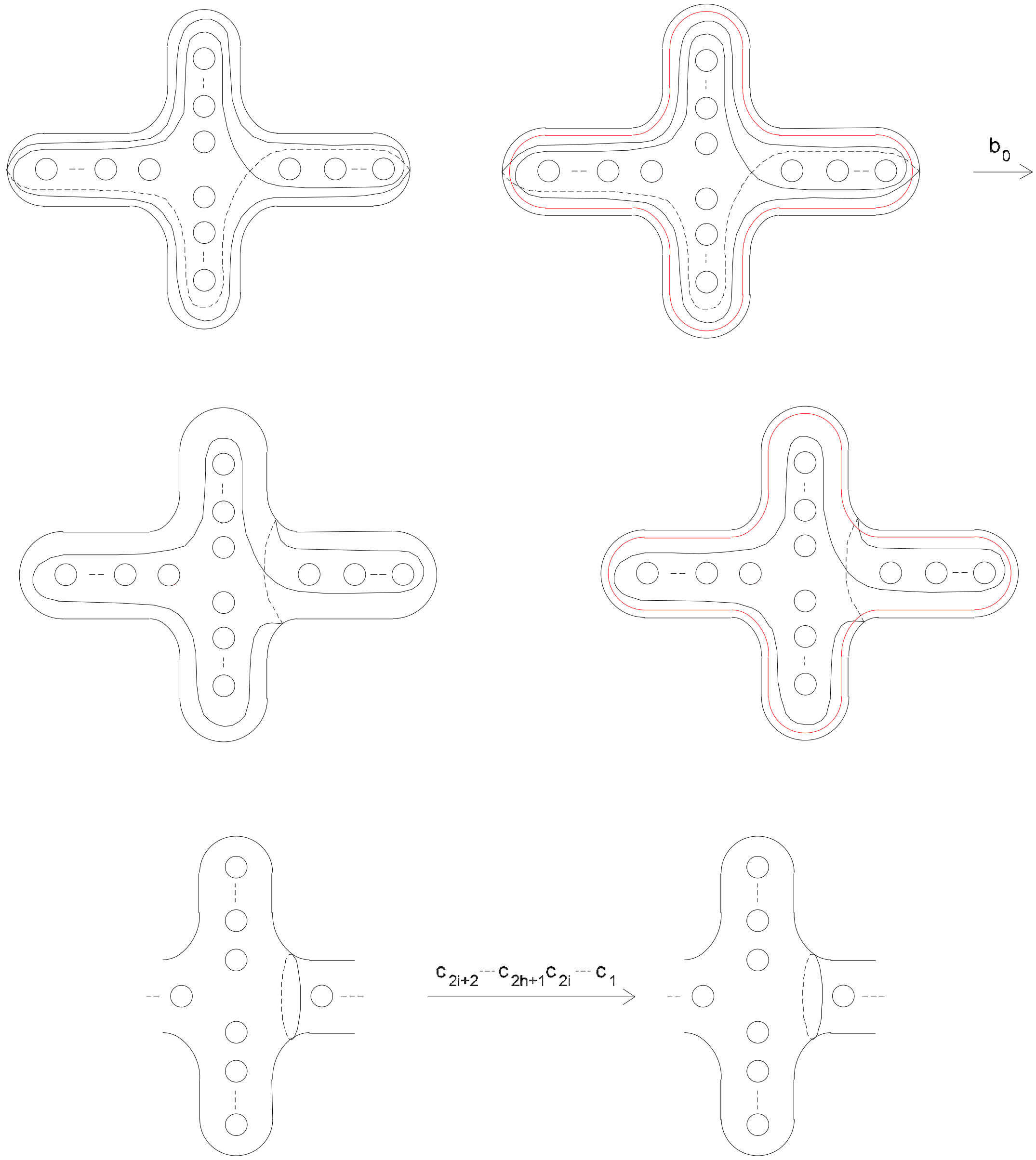


Figure 22. Mapping of a Separating Cycle

3. Applications

Next, we will compute the homeomorphism invariants of the genus g Lefschetz fibrations

$$X \longrightarrow S^2$$

described by the involution θ that was defined in Theorem 2.0.1, namely by the word $\theta^2 = 1$ in M_g, for some small values of g. Consider the surface in Figure 12. Let k denote the genus of the central part of the figure, which we will call the *vertical genus*. Let l be the total genus on the left and r be the total genus on the right of the vertical component. Let h be the *horizontal genus*, namely the sum of the *left genus* and the *right genus*; so, $h = l + r$. If we denote the total genus by g then $g = h + k = l + r + k$. A quick check reveals that the total number of twists in the word $\theta^2 = 1$ is $8h + 2k + 4 = 2g + 4 + 6h$.

Using the algorithm described in [17] we wrote a Matlab program that computes the signature of the manifold described by the word $\theta^2 = 1$ for given l, r, and k. The following table lists the output of the program for a few values of l, r, and k. The first column is for the input of the program in the format (l, k, r), the second column is for the *horizontal genus* $h = l + r$, the third column is for the *vertical genus* k, the fourth column is for the total genus $g = h + k$, the fifth column is for the word length $w = 8h + 2k + 4$, and the last column is for the output of the program, which is the signature of the manifold for the given triple (l, k, r).

(l,k,r)	h=l+r	k	g=h+k	w=8h+2k+4	signature
(1,2,1)	2	2	4	24	-12
(1,4,1)	2	4	6	28	-12
(1,6,1)	2	6	8	32	-12
(1,8,1)	2	8	10	36	-12
(1,10,1)	2	10	12	40	-12
(2,2,1)	3	2	5	32	-16
(1,2,2)	3	2	5	32	-16
(2,4,1)	3	4	7	36	-16
(1,4,2)	3	4	7	36	-16
(2,6,1)	3	6	9	40	-16
(1,6,2)	3	6	9	40	-16
(2,8,1)	3	8	11	44	-16
(1,8,2)	3	8	11	44	-16
(3,2,1)	4	2	6	40	-20
(2,2,2)	4	2	6	40	-20
(1,2,3)	4	2	6	40	-20
(3,4,1)	4	4	8	44	-20
(2,4,2)	4	4	8	44	-20
(1,4,3)	4	4	8	44	-20
(3,6,1)	4	6	10	48	-20
(2,6,2)	4	6	10	48	-20
(1,6,3)	4	6	10	48	-20
(3,8,1)	4	8	12	52	-20
(2,8,2)	4	8	12	52	-20
(1,8,3)	4	8	12	52	-20

The outputs consist of 0's and -1's only. The order in which they appear is the same for a fixed value of h and k in all the examples above. In other words the order in which the signature contributions of the $2-$ handles appear does not depend on how the horizontal genus is distributed to left and right, once we fix h and k. Therefore we will not record l and r for the rest of the examples.

h	k	g=h+k	w=8h+2k+4	signature
5	2	7	48	-24
5	4	9	52	-24
5	6	11	56	-24
5	8	13	60	-24
6	2	8	56	-28
6	4	10	60	-28
6	6	12	64	-28
6	8	14	68	-28
7	2	9	64	-32
7	4	11	68	-32
7	6	13	72	-32
8	2	10	72	-36
8	4	12	76	-36
8	6	14	80	-36

The signature values in the tables above are statistically convincing that the signature of the Lefschetz fibration given by the word $\theta^2 = 1$ must be

$$\sigma(X) = -4(h+1).$$

We do not know a topological proof for that; however, for a proof of this claim for $h = 0, 1$ the reader is referred to [9].

All of the Dehn twists appearing in the expression for θ are about nonseparating cycles. Therefore the Lefschetz fibration that is given by the word $\theta^2 = 1$ has

$$8h + 2k + 4$$

irreducible singular fibers and its Euler characteristic $\chi(X)$ is

$$\begin{aligned}\chi(X) &= 2(2-2g) + 8h + 2k + 4\\ &= 2(2-2(h+k)) + 8h + 2k + 4\\ &= 8 + 4h - 2k.\end{aligned}$$

For the values of the Euler characteristic and the signature above, $c_1^2(X)$ and $\chi_h(X)$ are

$$\begin{aligned}c_1^2(X) &= 3\sigma(X) + 2\chi(X)\\ &= 3(-4h-4) + 2(8+4h-2k)\\ &= -4h - 4k + 4\\ &= -4(g-1)\end{aligned}$$

and

$$\begin{aligned}
\chi_h(X) &= \frac{1}{4}(\sigma(X)+\chi(X)) \\
&= \frac{1}{4}(8+4h-2k-4h-4) \\
&= \frac{1}{4}(4-2k) = 1-k/2
\end{aligned}$$

$\chi_h(X)$ in the above computation makes sense because X has almost complex structure. It is an integer since k is even.

The following are the actual computer outputs for the signature computations for the indicated values of (l,k,r).

(l,k,r)	output
(1,2,1)	0 +0 +0 +0 +0 +0 -1 -1 -1 -1 -1 -1 -1 -1 +0 -1 -1 -1 -1 +0 +0 +0 +0 +0 = -12
(1,4,1)	0 +0 +0 +0 +0 +0 +0 +0 -1 -1 -1 -1 -1 -1 -1 -1 +0 +0 +0 -1 -1 -1 -1 +0 +0 +0 +0 +0 = -12
(1,6,1)	0 +0 +0 +0 +0 +0 +0 +0 +0 +0 -1 -1 -1 -1 -1 -1 -1 -1 +0 +0 +0 +0 +0 -1 -1 -1 -1 +0 +0 +0 +0 +0 = -12
(2,2,1)	0 +0 +0 +0 +0 +0 +0 +0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 +0 -1 -1 -1 -1 -1 -1 +0 +0 +0 +0 +0 +0 +0= -16
(2,2,2)	0 +0 +0 +0 +0 +0 +0 +0 +0 +0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 +0 -1 -1 -1 -1 -1 -1 -1 -1 +0 +0 +0 +0 +0 +0 +0 +0 +0 = -20
(2,4,2)	0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 +0 +0 +0 -1 -1 -1 -1 -1 -1 -1 -1 +0 +0 +0 +0 +0 +0 +0 +0 +0 = -20
(l,k,r)	output
(3,2,2)	0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 +0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0= -24
(4,2,4)	0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 +0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0 +0+0 +0 +0 +0 +0 +0 = -36

Acknowledgment I would like to express my deepest gratitude to my thesis advisor Ronald J. Stern for his continuous support and guidance.

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